European Girls' Mathematical Olympiad

2014- Turkey

Six Proposed Problems
of the Islamic Republic of Iran's team

by

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Problem 1: How many pairs (m, n) of integers are there such that

$$f(m, n) = 5m^2 - n^2 - 3m - n = 0$$

Solution: Since $f(1,1) = 0$ then $(1,1)$ is one of the desired pairs. We find integer coefficients a, b such that:

$$(an+bm+1)(5an+5bm+2)-(bn+5am+1)(bn+5am+2)=m(5m-3)-n(n+1)$$

We have: $a= -4$, $b= -9$.

If we put

$$(m_1, n_1) = (1,1) \text{ and } (m_{k+1}, n_{k+1}) = (-4n_k - 9m_k + 1, -9n_k - 20m_k + 1)$$

It is obvious that for every positive integer k,

$$f(m_k, n_k) = 0$$

Since

$$|m_{k+1}| > |m_k|$$

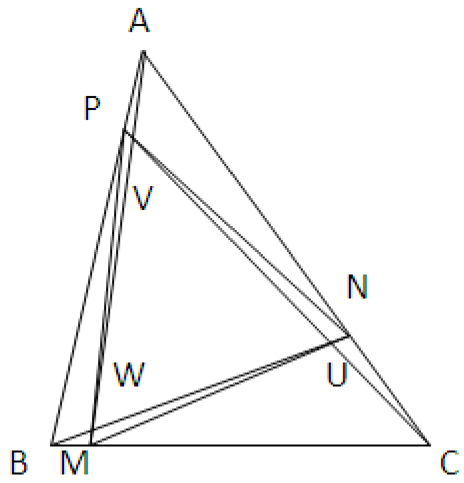
Thus there are infinitely many pairs (m, n) of integers such that $f(m, n)$=0.

Problem 2: Consider acute triangle ΔABC and points *M*, *N*, *P* on sides *BC*, *CA*, *AB* respectively such that

$$\angle ANM = \angle BPN = \angle CMP$$

Suppose that *(BN, PC)= U*, *(PC, AM)=V*, *(AM, BN)= W* (Notation for intersection two segments). Prove that the circumscribed circles of triangles ΔMNW, ΔNPU , and ΔPMV pass through a common point.

Solution:



By looking at the above figure,

$$\angle A = \angle PNM, \angle B = \angle MPN, \angle C = \angle PMN$$

Therefore $\Delta ABC \sim \Delta MNP$.

Now we prove some lemmas:

Lemma 1: The center of rotational homothety that maps segment AB to segment CD coincides with the center of rotational homothety that maps segment AC to segment BD.

Proof: Suppose that O is the center of rotational homothety that maps segment AB to segment CD. Thus $\angle AOB = \angle COD$ and $\frac{AO}{BO} = \frac{CO}{DO}$, hence $\angle AOC = \angle BOD$ and $\frac{AO}{CO} = \frac{BO}{DO}$ then $\Delta AOC \sim \Delta BOD$ and O is the center of rotational homothety that maps segment AC to segment BD.

Lemma 2: Suppose that M is the intersection point of two segments AB, CD. The circumscribed circles of triangles ΔMAC and ΔMBD pass through the center of rotational homothety that maps segment AB to segment CD.

Proof: Suppose that O is the center of rotational homothety that maps segment AB to segment CD. Since the oriented angle between MA and AO = the oriented angle between MC and CO, and also the oriented angle between MB and BO = the oriented angle between MD and DO, therefore the circumscribed circles of triangles ΔMAC and ΔMBD pass through point O and lemma 2 is proved. Suppose that O is the center of rotational homothety that maps ΔABC to ΔMNP. By using the above lemmas, the circumscribed circles of pairs of triangles ΔABW, ΔMNW - ΔBCU, ΔNPU - ΔCAV, ΔPMV pass through O. Thus the circumscribed circles of ΔMNW, ΔNPU , ΔPMV pass through a common point.

Problem 3: For a function $f\colon \mathbb{R} \to \mathbb{R}$ and a positive integer n, let $f^n = fofo \dots of$, *n*-th iterate of *f*. We say that f is *"nice"* , if for any two non empty open intervals *I* and *J* of ℝ, there exists *n* such that $f^n(I) \cap J \neq \emptyset$. $(f^n(I) = \{f^n(x): x \in I\})$

a) Find a nice function.
b) Is there a nice polynomial : $\mathbb{R} \to \mathbb{R}$?

Solution: a) For every integer k, we let $A_k = \{x \in \mathbb{R}: 2k \leq x \leq 2k+1\}$, $B_k = \{x \in \mathbb{R}: 2k-1 \leq x \leq 2k\}$ and define f in the following way:

For $x \in A_k$, we put $f(x) = -3x + 8k + 2$ and for $x \in B_k$, put $f(x) = 5x - 8k + 2$ **.**

For *n* even: $f(n, n+1) = (n-1, n+2), f(n-1, n+2) = (n-3, n+4)$, and so on.

For *n* odd: $f(n, n+1) = (n-2, n+3), f(n-2, n+3) = (n-4, n+5)$, and so on.

Thus f is a nice function.

b)No. Since the range of an even degree polynomial is a proper subset of ℝ , hence there is no nice even degree polynomial. Suppose that *P* is a nice odd degree polynomial. Therefore the following expression is satisfied:

There exists a real number a such that for any non empty open interval I, the following set has infinitely many members:

$$\{P^n(a): n \in \mathbb{N}\} \cap I$$

Case 1: deg(*P*)=1

We let $P(x) = bx + c$,

$$\{P^n(a): n \in \mathbb{N}\} = \{ba + c, b^2a + bc + c, \dots\}$$

If $|b| < 1$ then $lim_{n\to\infty}^{P^n(a)} = \frac{c}{1-b}$, if $|b| > 1$ then $lim_{n\to\infty}^{|P^n(a)|} = \infty$, if $b = 1$ then $P^n(a) = a + nc$, and finally if $b = -1$ then $\{P^n(a): n \in \mathbb{N}\}$ has two members. These results contradict the above expression.

Case 2: deg(P)≥ 3

Lemma: For a function $f\colon \mathbb{R} \to \mathbb{R}$, there is a positive real number m such that for any *x*, $|x| \geq m$ we have $|f(x)| \geq |x|$ and there exists an open interval *K*, such that $K \subset \{x: |f(x)| < |x|\}$, then f hasn't the property of the above expression.

Proof: We note that if $|P^n(a)| > m$ then $|P^l(a)| > m$ for any $l \geq n$.

By using this lemma, we conclude that there is no nice odd degree polynomial.

Thus there is no nice real polynomial.

Problem 4: Six students, Azer, Birgit, Charles, Dan, Emil, and Geoff from EGMOLAND are doing the following game:

In each move, they must choose six distinct positive integers *a, b, c, d, e, g*, respectively such that

$$M = abcdeg + (a+b)^2$$

be a perfect square. They aren't authorized to choose the numbers were used in the previous moves. Will the game get stopped?

Solution: No.

Lemma: For the Fibonacci sequence: $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$, the following identities are satisfied:

1- *Catalan's Identity* :

$$F_n F_{n+2r} + (-1)^n F_r^2 = F_{n+r}^2 \quad (r \text{ is a nonnegative integer})$$

2- *Generalized Catalan's Identity*:

$$F_n F_{n+r+s} + (-1)^n F_r F_s = F_{n+r} F_{n+s} \quad (r, s \text{ are nonnegative integers})$$

By using lemma ,

$$4F_n F_{n+1} F_{n+2} F_{n+4} F_{n+5} F_{n+6} + 4(3F_n + 4F_{n+1})^2 = (F_{n+1} F_{n+2} F_{n+6} + F_n F_{n+4} F_{n+5})^2$$

Since $F_{n+1} F_{n+2} F_{n+6}$ and $F_n F_{n+4} F_{n+5}$ are even, and $F_{n+2} + F_{n+4} = 3F_n + 4F_{n+1}$,

$$F_n F_{n+1} F_{n+2} F_{n+4} F_{n+5} F_{n+6} + (F_{n+2} + F_{n+4})^2 = \left(\frac{F_{n+1} F_{n+2} F_{n+6} + F_n F_{n+4} F_{n+5}}{2}\right)^2$$

Thus the game will not be stopped!

Problem 5: Let x, y, z be positive real numbers such that $xy+yz+zx=1$, prove the following inequality:

$$\frac{(1-x^2)(1-y^2)}{(1+x^2)(1+y^2)}+\frac{(1-y^2)(1-z^2)}{(1+y^2)(1+z^2)}+\frac{(1-z^2)(1-x^2)}{(1+z^2)(1+x^2)}\le\frac{3}{4}$$

Solution: First we note that if α, β, γ are angles of a triangle ABC ($\alpha+\beta+\gamma=180^{\circ}$), then:

$$tan\frac{\gamma}{2}=cotan\left(\frac{\alpha+\beta}{2}\right)=\frac{1-tan\frac{\alpha}{2}tan\frac{\beta}{2}}{tan\frac{\alpha}{2}+tan\frac{\beta}{2}}$$

$$\Rightarrow tan\frac{\alpha}{2}tan\frac{\beta}{2}+tan\frac{\beta}{2}tan\frac{\gamma}{2}+tan\frac{\gamma}{2}tan\frac{\alpha}{2}=1$$

Therefore we put $x=tan\frac{\alpha}{2}, y=tan\frac{\beta}{2}, z=tan\frac{\gamma}{2}$ and the inequality is written in the following form:

$$\frac{\left(1-tan^2\frac{\alpha}{2}\right)(1-tan^2\frac{\beta}{2})}{\left(1+tan^2\frac{\alpha}{2}\right)(1+tan^2\frac{\beta}{2})}+\frac{\left(1-tan^2\frac{\beta}{2}\right)(1-tan^2\frac{\gamma}{2})}{\left(1+tan^2\frac{\beta}{2}\right)(1+tan^2\frac{\gamma}{2})}+\frac{\left(1-tan^2\frac{\gamma}{2}\right)(1-tan^2\frac{\alpha}{2})}{\left(1+tan^2\frac{\gamma}{2}\right)(1+tan^2\frac{\alpha}{2})}\le\frac{3}{4}$$

or equivalently

$$cos\alpha\ cos\beta+cos\beta\ cos\gamma+cos\gamma\ cos\alpha\le\frac{3}{4}$$

Now we prove some lemmas:

Lemma 1: $sin\frac{\alpha}{2}\ sin\frac{\beta}{2}\ sin\frac{\gamma}{2}=\frac{r}{4R}$, where r, R are radii of circumscribed and inscribed circles of ΔABC, respectively.

Proof: By constructing the inscribed circle of ΔABC,

$$2Rsin\gamma=r\left(cotan\frac{\alpha}{2}+cotan\frac{\beta}{2}\right)=\frac{rsin(\frac{\alpha+\beta}{2})}{sin\frac{\alpha}{2}sin\frac{\beta}{2}}=\frac{rcos\frac{\gamma}{2}}{sin\frac{\alpha}{2}sin\frac{\beta}{2}}$$

By using $sin\gamma=2sin\frac{\gamma}{2}cos\frac{\gamma}{2}$, lemma 1 is proved.

Lemma 2: $cos\alpha+cos\beta+cos\gamma\le\frac{3}{2}$.

Proof:

$$cos\alpha+cos\beta=2cos\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}$$

$$cos\gamma = -\cos(\alpha+\beta) = -2cos^2\frac{\alpha+\beta}{2} + 1$$

$$cos\frac{\alpha-\beta}{2} - cos\frac{\alpha+\beta}{2} = 2sin\frac{\alpha}{2}sin\frac{\beta}{2}$$

From the above equalities and lemma 1,

$$cos\alpha + cos\beta + cos\gamma = 4sin\frac{\alpha}{2}\ sin\frac{\beta}{2}\ sin\frac{\gamma}{2} + 1 = \frac{r}{R} + 1$$

By using the Euler's inequality ($2r \leq R$) lemma 2 is proved.

Lemma 3: $cos^2\alpha + cos^2\beta + cos^2\gamma \geq \frac{3}{4}$.

Proof:

$$cos2\alpha + cos2\beta = 2\cos(\alpha+\beta)\cos(\alpha-\beta) = -2cos\gamma\ \cos(\alpha-\beta)$$

$$cos2\gamma = 2cos^2\gamma - 1 = -2cos\gamma\cos(\alpha+\beta) - 1$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2cos\alpha\ cos\beta$$

From the above equalities,

$$cos2\alpha + cos2\beta + cos2\gamma + 4cos\alpha\ cos\beta\ cos\gamma + 1 = 0$$

By using $cos2x = 2cos^2x - 1$,

$$cos^2\alpha + cos^2\beta + cos^2\gamma = 1 - 2cos\alpha\ cos\beta\ cos\gamma\ \ (*)$$

For an acute triangle ΔABC, by the AM-GM inequality and lemma 2,

$$cos\alpha\ cos\beta\ cos\gamma \leq (\frac{cos\alpha + cos\beta + cos\gamma}{3})^3 \leq \frac{1}{8}\quad (**)$$

For an obtuse triangle ΔABC,

$$cos\alpha\ cos\beta\ cos\gamma < 0$$

Lemma 3 is derived from (*), (**).

Finally, by using the following identity and three lemmas, problem 5 is solved.

$$cos\alpha\ cos\beta + cos\beta\ cos\gamma + cos\gamma\ cos\alpha = \frac{1}{2}[(cos\alpha + cos\beta + cos\gamma)^2 - (cos^2\alpha + cos^2\beta + cos^2\gamma)]$$

Problem 6: Consider the sequence *1, 2, …, n* and in each move invert every maximal increasing subsequence to reach the arrangement *n, n-1, …, 2, 1* . Prove that we reach new arrangement in at most *n-1* moves.

Solution: We prove that for any *k*, the numbers aren't greater than *k*, will be to the right of the numbers *k+1, k+2, …, n* after at most *n-1* moves. Let *A* and *B* be these sets respectively, and suppose that $\mathcal{M}_0$ is the *A-B* pattern for the initial arrangement and $\mathcal{M}_i$ is the arrangement after move *i* . Also let $\mathcal{N}_0$ be *aa…abb…b* (*k a*'s and *n-k b*'s) and $\mathcal{N}_i$ be obtained from $\mathcal{N}_{i-1}$ by changing each *ab* into *ba*. It is clear that $\mathcal{N}_{n-1}$ is *bb…baa…a* .

We claim that the *j*-th *b* (from the left) in $\mathcal{M}_i$ is no farther from the left end than the *j*-th *b* in $\mathcal{N}_i$. It is clear for *i=0.* Suppose that it is true for *i-1* and we prove it is true for *i* .

For *j=1:* in $\mathcal{M}$'s the leftmost *b* moves to the left by at least 1 in each move until it reaches the left-end position and in $\mathcal{N}$'s it moves to the left by exactly 1 position in each move until it reaches the left-end position. Let *j* be the smallest value, if any, for which it fails. The *j*-th *b* in $\mathcal{M}_{i-1}$ didn't move then in move *i* , whereas the *j*-th *b* in $\mathcal{N}_{i-1}$ moved by one place. Since the *j*-th *b* in $\mathcal{M}_{i-1}$ is not to the right of the *j*-th *b* in $\mathcal{N}_{i-1}$ they must be in the same place in these two words. But the *j*-th *b* in $\mathcal{M}_{i-1}$ doesn't move in move *i* , it must be immediately preceded by the *j-1* -th *b,* while in $\mathcal{N}_{i-1}$ it is preceded by an *a* . Then the *j-1* -th *b* in $\mathcal{M}_{i-1}$ is to the right of the *j-1* -th *b* in $\mathcal{N}_{i-1}$, contrary to assumption.

Hence we need at most *n-1* moves to reach *n, n-1, …, 2, 1* .